Student Number

2022 GLENWOOD HIGH SCHOOL

Trial Higher School Certificate Examination

Mathematics Extension 1

General Instructions

* Reading Time – 10 minutes
* Working time – 2 hours
* Write using black pen
* NESA approved calculators may be used
* A reference sheet is provided
* For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 4)

* Attempt Questions 1-10
* Allow about 15 minutes for this section

Section II – 60 marks (pages 5 – 11)

* Attempt Questions 11 – 14
* Allow about 1 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

1. A vector $\underset{\sim}{v}$ is defined as $\underset{\sim}{v} = 6\underset{\sim}{\imath} - 8\underset{\sim}{\jmath}$. What is a unit vector in the direction of $\underset{\sim}{v}$?

   A. $\hat{\underset{\sim}{u}} = 3\underset{\sim}{\imath} - 4\underset{\sim}{\jmath}$

   B. $\hat{\underset{\sim}{u}} = 1\underset{\sim}{\imath} - 1\underset{\sim}{\jmath}$

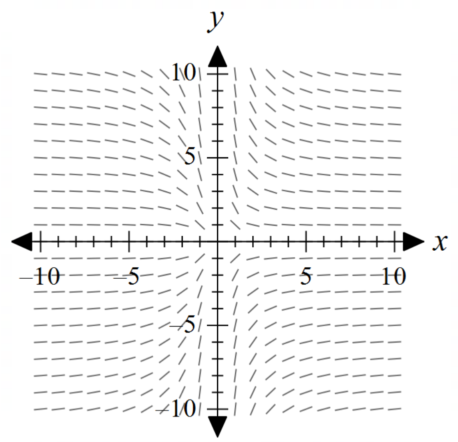
   C. $\hat{\underset{\sim}{u}} = \frac{3}{5}\underset{\sim}{\imath} - \frac{4}{5}\underset{\sim}{\jmath}$

   D. $\hat{\underset{\sim}{u}} = \frac{3}{4}\underset{\sim}{\imath} - \frac{4}{3}\underset{\sim}{\jmath}$

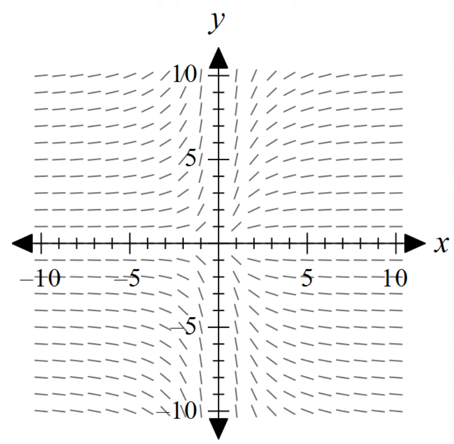
2. A differential equation is given to be $\dfrac{dy}{dx} = \dfrac{y}{x^2}$

   Which of the following best represents the direction field of the differential equation?

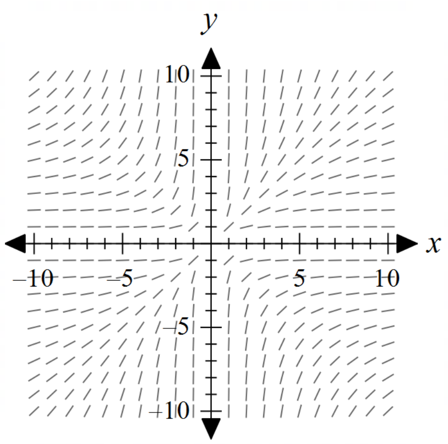
A.



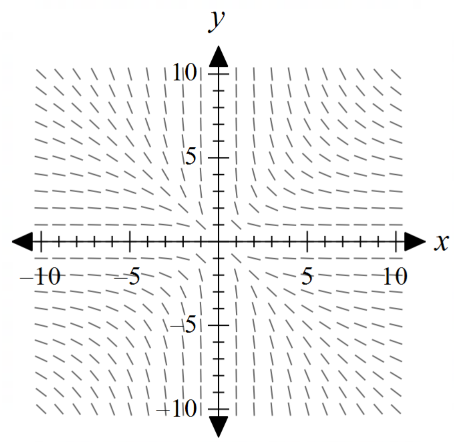
B.



C.



D.



3. Find the value of k such that $\int_0^k \frac{1}{4+x^2}\,dx = \frac{\pi}{6}$

A. 1

B. $\frac{1}{2}$

C. $\sqrt{3}$

D. $2\sqrt{3}$

4. How many solutions does the equation $\sin 6x - \sin 2x = 0$ have for $0 \leq x \leq 2\pi$?

A. 5

B. 12

C. 14

D. 13

5. Consider the function $f(x) = x^3 + x + 8$.

Which of the following is the point of intersection of the function $f(x)$ and its inverse $f^{-1}(x)$?

A. $(-1, 6)$

B. $(-2, -2)$

C. $(0, 0)$

D. $(2, 2)$

6. The equation $x^3 - 8x^2 - 4x + 2 = 0$ has roots α, β and γ. What is the value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$?

A. -4

B. -2

C. 2

D. 4

7. The equation $y = e^{mx}$ satisfies the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$.

What are the possible values of m?

A. $m = -2 \text{ or } m = 3$

B. $m = 2 \text{ or } m = 3$

C. $m = -2 \text{ or } m = -3$

D. $m = 2 \text{ or } m = -3$

8. The probability that it will rain on any given day in February 2024 is 0.2.

What is the probability that February 2024 (a leap year) will have exactly 8 rainy days?

A. 1.5×10^{-7} %

B. 9 %

C. 10 %

D. 29 %

9. What is the domain and range of the function $y = 6\cos^{-1}(3x)$?

A. Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[0, 6\pi]$.

B. Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[0, 3\pi]$.

C. Domain $[0, 6\pi]$; Range $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

D. Domain $[0, 3\pi]$; Range $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

10. A circular metal plate is heated so that its diameter is increasing at a constant rate of 0.005 m/s.

At what rate is the area of the circular surface of the plate increasing when its diameter is 6 metres?

A. 0.015π m^2/s

B. 0.038π m^2/s

C. 0.06π m^2/s

D. 0.075π m^2/s

End of Section I

Section II

60 marks

Attempt Questions 11 – 14.

Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new writing booklet.

(a) Given $(x+3)$ is a factor of $P(x) = x^3 + ax^2 - 7x + 6$, find the value of a and the other factors. **3**

(b) Differentiate: $x\sin^{-1}x + \sqrt{1-x^2}$ **2**

Hence evaluate $\int_0^{1/2} \sin^{-1} x \; dx$ **2**

(c) The ampere (or amp) is a unit used to measure electric current.
The current (i) in amperes, at time t, in a circuit is calculated using the equation

$$i = 12\sin t + 5\cos t$$

(i) Write an expression for i in the form $R\sin(t+\alpha)$ where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. Give the value of α to two decimal places. **2**

(ii) Using the result in part (i) or otherwise, find the maximum current in the circuit and the first time it occurs. Give your answer to two decimal places. **2**

(d) In a large population of birds, the proportion of cockatoos is $\frac{1}{5}$ **2**
Let $\hat{p}$ be the random variable that represents the sample proportion of cockatoos for samples of size n drawn from the population.
Find the smallest integer value of n such that the standard deviation of $\hat{p}$ is less than or equal to 1%. You may use $\sigma(\underset{\sim}{\hat{p}}) = \sqrt{\frac{p(1-p)}{n}}$

(e) Find the coefficient of x^3 in the expansion of $\left(2x - \frac{3}{x^2}\right)^9$ **2**

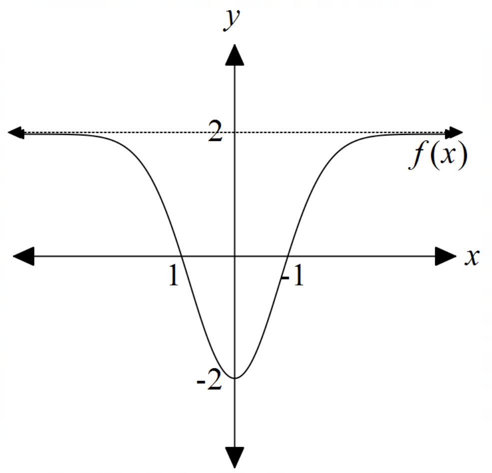
End of Question 11

Question 12 (15 marks) Use a new writing booklet.

(a) Suppose $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and θ is the acute angle between them. **4**
Calculate that the exact value of $\sin 4\theta$. Give clear reasoning for your answer.

(b) Use the principle of mathematical induction to show that for all integers **3**
$n \geq 1,\ 3 \times 5^{2n+1} + 2^{3n+1}$ is divisible by 17.

(c) The diagram shows the graph of $f(x)$, which has a y-intercept at -2, **2**
x-intercepts at 1 and -1, and a horizontal asymptote at $y = 2$.



Sketch a half – page sketch of $y = \dfrac{1}{f(x)}$, showing any asymptotes and intercepts.

(d) Solve the differential equation below, to give an equation for y in terms of x, given that $y\ (0) = 0$. **3**

$$\frac{dy}{dx} = e^x \cos^2 y$$

(e) Given $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$, **3**

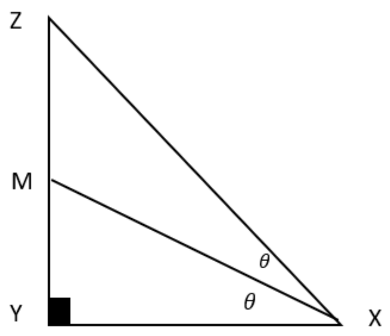
Use the substitution $u = \sqrt{x-1}$ to find the exact value of

$$\int_2^5 \frac{x-1}{x-1+\sqrt{x-1}}$$

End of Question 12

Question 13 (15 marks) Use a new writing booklet.

(a) ΔXYZ is a right-angled triangle, with M located along the side YZ as shown. **3**
The lengths of YZ and YX are 2 cm and $\sqrt{2}$ cm respectively. It is also known that $\angle MXY = \angle ZXM = \theta$.



Show that MY is $-1 + \sqrt{3}$

(b) Eight students, including Ari, Bobbi, and Cali, form a single file line and walk into their classroom.

(i) How many ways can the students walk into the classroom if Ari and Bobbi are next to each other, with no students in between? **1**

(ii) How many ways can the students walk into the classroom if Ari enters before Bobbi? **1**

(iii) When the students enter the classroom, they are seated around a circular table. **1**

How many ways can the students be seated around the table if Bobbi and Cali cannot sit next to each other?

Question 13 continues on page 8

(c) A 500 L tank contains 200 L of brine (salt in water) with 50 kg of salt dissolved.

Pure water is pumped into the tank at 20 L/min. At the same time, the perfectly mixed brine in the tank is pumped out of the tank at 15 L/min.

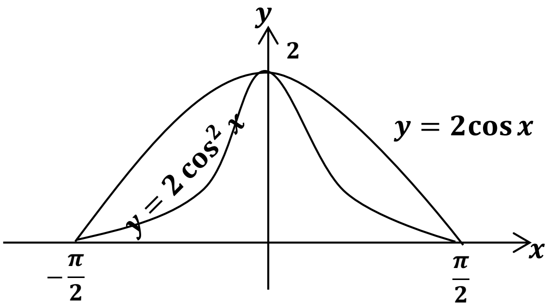
(i) Explain why the amount of salt m kg in the tank after t minutes can be modelled by the differential equation **1**

$$\frac{dm}{dt} = -\frac{3m}{40 + t}$$

(ii) Hence, find m as a function of t. **3**

(iii) How many kilograms of salt is in the tank when it begins to overfill? Give your answer correct to one decimal place. **2**

(d) Find the area between the curves $y = 2cosx$ and $y = 2cos^2x$ from $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ **3**



End of Question 13

Question 14 (15 marks) Use a new writing booklet.

(a) A biased coin which produces heads for 60% of tosses is to be tested. The coin is tossed 50 times.

(i) Find the expected value and variance of the number of heads. **1**

(ii) Use the normal distribution to approximate the probability that the coin comes up heads 23 to 27 times. **3**

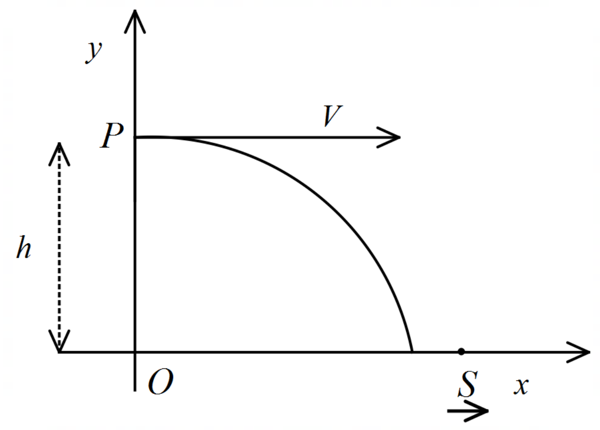
The standard normal table below gives $P(Z \leq z)$ for z scores from -2.20 to 0.09.

	Second Decimal Place									
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.2	0.0139	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174
-2.1	0.0179	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222
-2.0	0.0228	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281
-1.9	0.0287	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351
-1.9	0.0287	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351
-1.8	0.0359	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436
-1.7	0.0446	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537
-1.6	0.0548	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655
-1.5	0.0668	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793
-1.4	0.0808	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951
-1.3	0.0968	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131
-1.2	0.1151	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335
-1.1	0.1357	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562
-1.0	0.1587	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814
-0.9	0.1841	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090
-0.8	0.2119	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389
-0.7	0.2420	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709
-0.6	0.2743	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050
-0.5	0.3085	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409
-0.4	0.3446	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783
-0.3	0.3821	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168
-0.2	0.4207	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562
-0.1	0.4602	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359

(iii) Comment on the experiment's design to estimate the coin's bias and how it can be improved. **1**

Question 14 continues on page 10 and 11

(b) A particle is projected horizontally from a point P, h metres above O, with a velocity of V metres per second. The only acceleration that the particle undergoes is the acceleration due to gravity $(-g)$ vertically in a downward direction. There is no horizontal acceleration.



(i) Find $\underset{\sim}{s}$ the displacement vector of the particle at time t. **2**

(ii) A package of supplies is dropped from a plane to a disabled sailboat (S) in the ocean. The plane is travelling at a constant velocity of 252 km/h and is 150 m above sea level and directly due west of the disabled sailboat. **2**

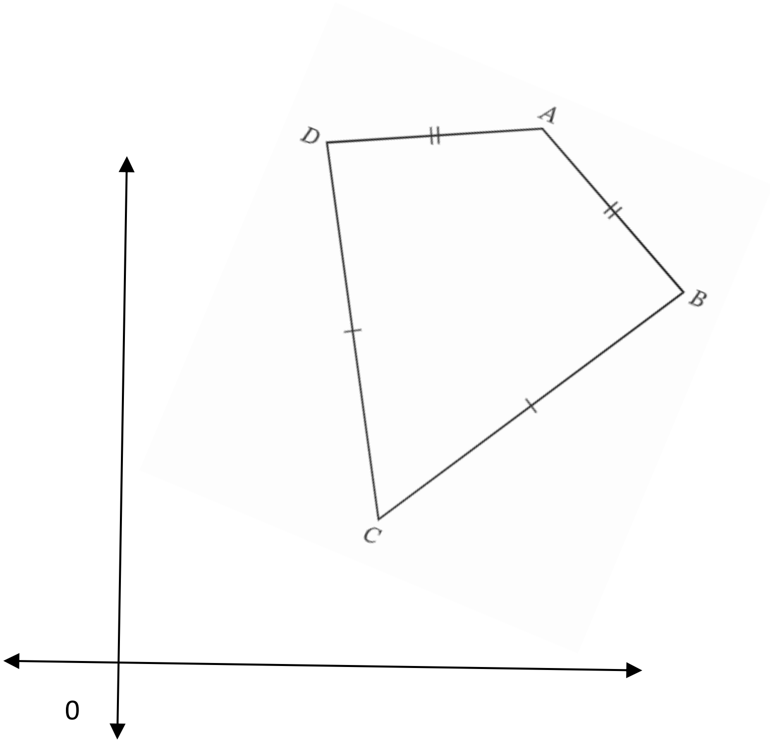
How long will it take the package to hit the water? (Take $g = 10$ m/s^2)

(iii) A current is causing the sailboat to drift at a speed of 1.8 km/h in the same direction the plane is travelling. What is the minimum and maximum horizontal distance between the plane and the sailboat when the package is dropped so that it lands at most 40m from the disabled sailboat? **3**

Question 14 continues on page 11

(c) Show that the diagonals of a kite are perpendicular, noting that lengths of $AB = AD$ and $BC = CD$. **3**

You are given $\overrightarrow{AB} = (b - a), \overrightarrow{BC} = (c - b), \overrightarrow{DC} = (c - d)$ and $\overrightarrow{AD} = (d - a)$



End of Examination

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C , B , D , D , B , A , A , C , A , A

1). $\hat{\underset{\sim}{u}} = \dfrac{6\underset{\sim}{i} - 8\underset{\sim}{j}}{\sqrt{6^2+8^2}} =$

$= \frac{1}{10}(6\underset{\sim}{i} - 8\underset{\sim}{j})$ (C)

$= \frac{3}{5}\underset{\sim}{i} - \frac{4}{5}\underset{\sim}{j}$

2). (B)

3). $\int_0^k \frac{1}{4+x^2}\,dx = \frac{1}{2}\left[\tan^{-1} x/2\right]_0^k$

$\frac{1}{2}\left[\tan^{-1}\frac{k}{2} - \tan^{-1}0\right] = \frac{\pi}{6}$

$\tan^{-1} k/2 = \frac{\pi}{\cancel{6}_3} \times \cancel{2} + \tan^{-1}0$

$\tan^{-1} k/2 = \frac{\pi}{3}$ (D)

$\frac{k}{2} = \tan\frac{\pi}{3}$

$k = 2\sqrt{3}$

4). $\sin 6x - \sin 4x$

$A + B = 6x$

$A - B = 4x$

$2A = 10x$

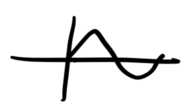
$A = 5x$

$B = x$ (D)

$\therefore \sin 6x - \sin 4x = 0$

$\cos 5x \sin x = 0$

$\cos 5x = 0$	$\sin x = 0$
10 solutions	3 solution.



5). $f(x)$ & $f^{-1}(x)$ meet on $y = x$

$$x^3 + x + 8 = x$$
$$x^3 + 8 = 0$$
$$x^3 = -8$$
$$x = -2$$
$$y = -2$$

B

6). $$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma}$$

$$\alpha\beta\gamma = -2$$
$$\alpha + \beta + \gamma = 8$$

A

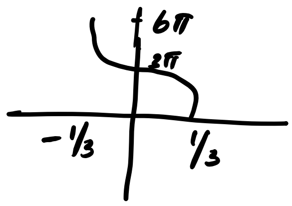
$$\therefore \quad \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{8}{-2} = -4$$

7). $$y = e^{mx}$$
$$\frac{dy}{dx} = me^{mx}$$
$$\frac{d^2y}{dx^2} = m^2 e^{mx}$$

A

$$m^2 e^{mx} - me^{mx} - 6e^{mx} = 0$$
$$m^2 - m - 6 = 0$$
$$(m-3)(m+2) = 0$$
$$m = 3, -2$$

8). $\binom{29}{8}(0.8)^{21}(0.2)^8 = 0.101 \approx 10\%$ C

9) 

$x \in [-1/3, 1/3]$
$y \in [0, 6\pi]$ (A)

10) $A = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi}{4}D^2$

$\frac{dA}{dD} = \frac{\pi}{\cancel{4}_2} \times \cancel{2}D = \frac{\pi D}{2}$

$\frac{dA}{dt} = \frac{dA}{dD} \times \frac{dD}{dt}$

$= \frac{\pi D}{2} \times 0.005$

when $D = 6$

$\frac{dA}{dt} = \frac{\pi \times 6}{2} \times 0.005$ (A)

$= 0.015\pi$

Question 11

a). $P(-3) = 0$

$(-3)^3 + a(-3)^2 - 7(-3) + 6 = 0$

$-27 + 9a + 21 + 6 = 0$

$9a = 0$

$a = 0$

$$
\begin{array}{r}
x^2 - 3x + 2 \\
x+3 \overline{\smash{\big)}\, x^3 + 0x^2 - 7x + 6} \\
\underline{x^3 + 3x^2} \\
-3x^2 - 7x \\
\underline{-3x^2 - 9x} \\
2x + 6 \\
2x + 6
\end{array}
$$

Using long division

$P(x) = (x+3)(x^2 - 3x + 2)$
$= (x+3)(x-2)(x-1)$

b). $\frac{d(x\sin^{-1}x+\sqrt{1-x^2})}{dx} = x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{1}{\cancel{2}\sqrt{1-x^2}} \times -\cancel{2}x$

$= \cancel{\frac{x}{\sqrt{1-x^2}}} + \sin^{-1}x - \cancel{\frac{x}{\sqrt{1-x^2}}}$

$= \sin^{-1}x$

$\int_0^{1/2} \sin^{-1}x = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{1/2}$

$= \frac{1}{2}\sin^{-1}(1/2) + \sqrt{1-1/4} - 0 - \sqrt{1-0^2}$

$= \frac{1}{2} \times \frac{\pi}{6} + \sqrt{\frac{3}{4}} - 1$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

c). $i = 12\sin t + 5\cos t$

$12\sin t + 5\cos t = R\sin(t+\alpha)$

$12\sin t + 5\cos t = R\sin t\cos\alpha + R\cos t\sin\alpha$

Equating coefficients of $\sin t$ & $\cos t$

$R\cos\alpha = 12$ —— ①

$R\sin\alpha = 5$ —— ②

$①^2 + ②^2 \qquad R^2 = 12^2 + 5^2$

$R = 13$

$② \div ① \qquad \alpha = \tan^{-1}(5/12)$

$= 0.39$

$\therefore \; 12\sin t + 5\cos t = 13\sin(t + 0.39)$

Max value occurs when $i = 13$

$$13\sin(t + 0.39) = 13$$

$$\sin(t + 0.39) = 1$$

$$t + 0.39 = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$t = \frac{\pi}{2} - 0.39,$$

$$= 1.18\,s$$

d). $\sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$

$$\sqrt{\frac{\frac{4}{5}\left(\frac{4}{5}\right)}{n}} \leq 0.01$$

$$\sqrt{\frac{4}{25n}} \leq 0.01$$

$$\frac{4}{25n} \leq (0.01)^2$$

$$\frac{4}{25(0.01)^2} \leq n$$

$$1600 \leq n$$

$$n \geq 1600$$

e). $\left(2x - \frac{3}{x^2}\right)^9$

$$T_{k+1} = \binom{9}{k}(2x)^{9-k}\left(\frac{-3}{x^2}\right)^k$$

$$= \binom{9}{k} 2^{9-k}(-3)^k x^{9-k}(x^{-2})^k$$

$$= \binom{9}{k} 2^{9-k}(-3)^k x^{9-3k}$$

$$9-3k = 3$$

$$-3k = -6$$

$$k = 2$$

coeffient of T_3 $= \binom{9}{2} 2^7 (-3)^2$

$= 41472$

Question 12

12a) $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \underset{\sim}{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

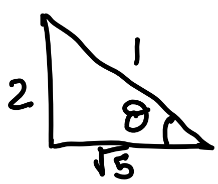
$$\cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$

$$= \frac{2 \times 0 + 3 \times 1}{(\sqrt{2^2+1^2})(\sqrt{3^2})}$$

$$= \frac{\cancel{3}}{\sqrt{5} \times \cancel{3}}$$

$$= \frac{1}{\sqrt{5}}$$

As θ is acute (given)



$\therefore \quad \sin\theta = \frac{2}{\sqrt{5}}$

As $\quad \sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}}$$

$$\sin 2\theta = \frac{4}{5}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \left(\frac{1}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2$$

$$= \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$$

$\therefore \quad \sin 4\theta = 2\sin 2\theta \cos 2\theta$

$\sin 4\theta = 2 \times \frac{4}{5} \times \left(-\frac{3}{5}\right)$

$\sin 4\theta = -\frac{24}{25}$

12b) $3 \times 5^{2n+1} + 2^{3n+1}$ divisible by 17, $n \geq 1$

Check true for $n=1$

$3 \times 5^{2+1} + 2^{3+1} = 3 \times 5^3 + 2^4$

$= 391$

$= 17 \times 23$

$\therefore$ true for $n=1$

Assume true for k, where $k \in \mathbb{Z}^+$

i.e. $3 \times 5^{2k+1} + 2^{3k+1} = 17P$ —— (1)

where $P \in \mathbb{Z}$

Prove true for $k+1$

i.e. $3 \times 5^{2(k+1)+1} + 2^{3(k+1)+1} = 17Q$, where $Q \in \mathbb{Z}$

$L.H.S = 3 \times 5^{2k+2+1} + 2^{3k+3+1}$

$= 3 \times 5^{2k+1}(5^2) + 2^{3k+1}(2^3)$

$= 3 \times 25 \times 5^{2k+1} + 8 \times 2^{3k+1}$

$= 3 \times 25 \times 5^{2k+1} + 25 \times 2^{3k+1} - 17 \times 2^{3k+1}$

$= 25(3 \times 5^{2k+1} + 2^{3k+1}) - 17 \times 2^{3k+1}$

substituting equation (1)

$LHS = 25 \times 17P - 17 \times 2^{3k+1}$

$= 17(25P - 2^{3k+1})$

As P and $K \in \mathbb{Z}$

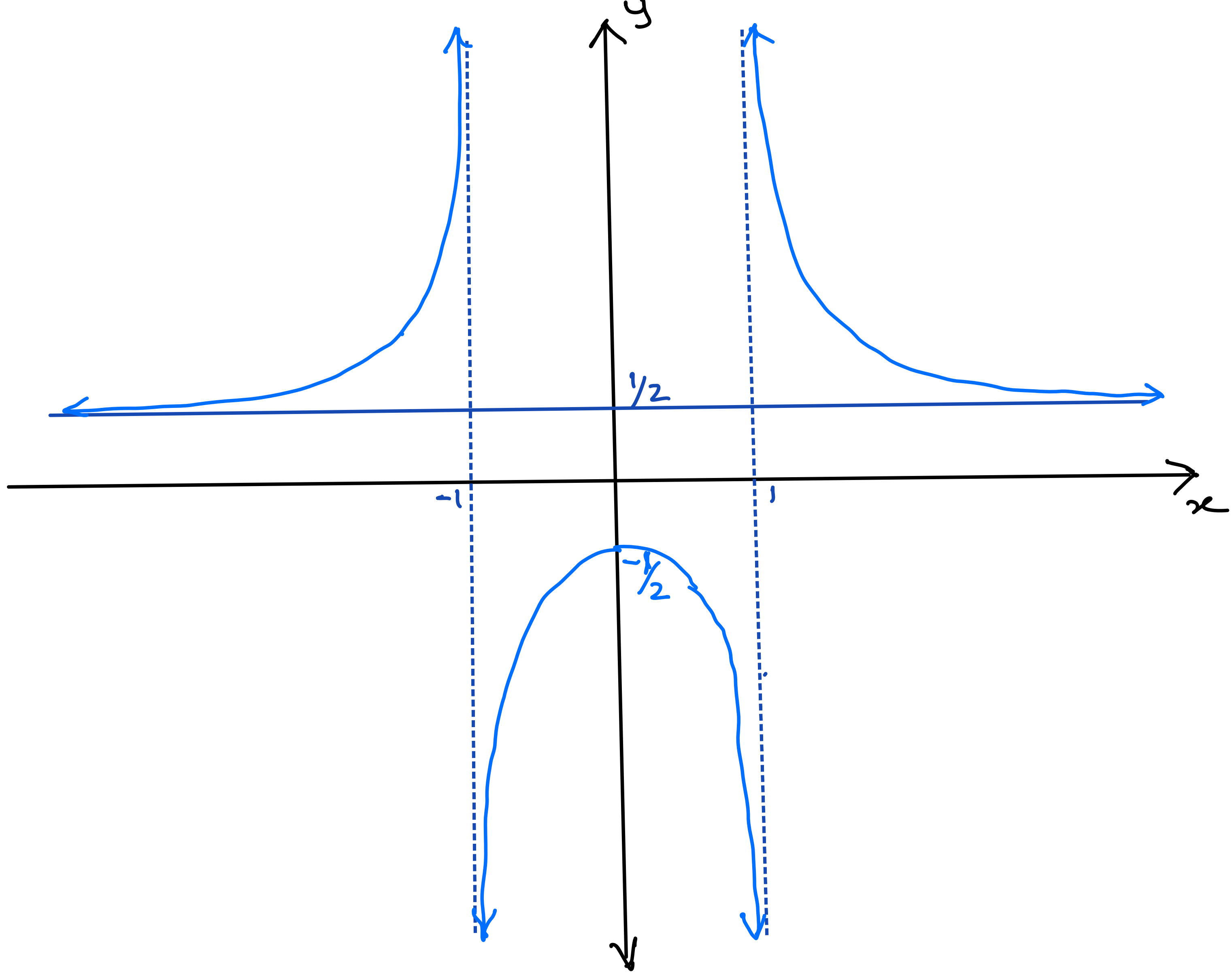
$\therefore \ 25P - 2^{3K+1} \in \mathbb{Z}$

$\therefore \ L.H.S = 17Q$

$\therefore$ True for $k+1$

Hence, by mathematical induction $3 \times 5^{2n+1} + 2^{3n+1}$ is true for all positive integers $n \geq 1$

12c)



12d)

$$\frac{dy}{dx} = e^x \cos^2 y$$

$$\frac{dy}{\cos^2 y} = e^x \, dx$$

$$\sec^2 y \, dy = e^x \, dx$$

Integrating both sides

$$\int \sec^2 y \, dy = \int e^x \, dx$$

$$\tan y = e^x + c$$

Given $y(0) = 0$ condition

$$\tan(0) = e^0 + c$$

$$0 = 1 + c$$

$$\therefore c = -1$$

$$\therefore \tan y = e^x - 1$$

$$y = \tan^{-1}(e^x - 1)$$

12e)

$$\int_2^5 \frac{x-1}{x-1+\sqrt{x-1}} \, dx$$

Given $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$

Using substitution $u = \sqrt{x-1}$

$u = \sqrt{x-1}$

$\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x-1}} \times (1)$

$\dfrac{dx}{du} = 2\sqrt{x-1}$

$\therefore dx = 2\sqrt{x-1}\, du$

$= 2u\, du$

when $x = 2$

$u = \sqrt{2-1} = 1$

when $x = 5$

$u = \sqrt{5-1} = \pm 2$

taking $u = 2$ as the given integral is positive.

$\therefore \displaystyle\int_1^2 \frac{u^2}{u^2+u} \times 2u\, du$

$= \displaystyle\int_1^2 \frac{2u^3}{u(u+1)}\, du$

$= 2\displaystyle\int_1^2 \frac{u^2}{u+1}\, du$

using the given equation

$= 2\displaystyle\int_1^2 \left(u - 1 + \frac{1}{u+1}\right) du$

$= 2\left[\frac{u^2}{2} - u + \ln(u+1)\right]_1^2$

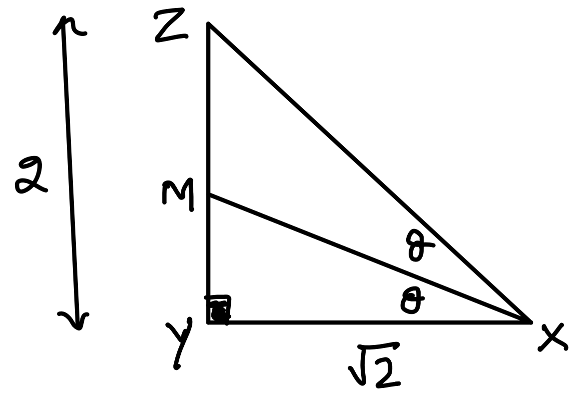
$= 2\left(\frac{2^{\cancel{2}}}{\cancel{2}} - 2 + \ln 3 - \frac{1}{2} + 1 - \ln 2\right)$

$= 2\left(\frac{1}{2} + \ln\frac{3}{2}\right)$

$= 1 + 2\ln\frac{3}{2}$

Question 13

13a)



$$\text{Tan } 2\theta = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\therefore \quad \frac{2\tan\theta}{1-\tan^2\theta} = \sqrt{2}$$

$$2\tan\theta = \sqrt{2} - \sqrt{2}\tan^2\theta$$

$$\sqrt{2}\tan^2\theta + 2\tan\theta - \sqrt{2} = 0$$

$$\tan^2\theta + \sqrt{2}\tan\theta - 1 = 0$$

$$\tan\theta = \frac{-\sqrt{2} \pm \sqrt{2+4}}{2}$$

$$= -\frac{\sqrt{2} \pm \sqrt{6}}{2}$$

As θ is acute

$$\therefore \quad \tan\theta = \frac{-\sqrt{2} + \sqrt{6}}{2}$$

From the diagram

$$\tan\theta = \frac{MY}{\sqrt{2}}$$

$$\therefore \quad MY = \sqrt{2}\left(\frac{\sqrt{6} - \sqrt{2}}{2}\right) = \frac{\sqrt{2}(\sqrt{2}\sqrt{3} - \sqrt{2})}{2}$$

$$= \sqrt{3} - 1 \quad \text{or} \quad -1 + \sqrt{3}$$

13b) i) $7! \times 2! = 10080$ as Ari & Bobbi are 1 unit with 2! arrangements

ii) $\frac{8!}{2} = 20160$ divided by 2 for ways when Bobbi enters before Ari

iii) 7! for round table

$7! = 5040$

If Bobby & Cali sit together and form a unit, then arragements

$(7-1)! \times 2! = 1440$

$\therefore$ Not next to each other

$= 5040 - 1440$

$= 3600$

13c)

Tank 500L

Brine 200L with 50 kg salt

Pure water pump in 20L/min

Brine pump out 15L/min

Adding 5L/min

Volume at any time 't' $= 200 + 5t$

i) $\frac{dm}{dt}$ (Kg/min) = Change in mass rate of salt

= mass rate pumped in − mass rate pumped out

$\therefore$ $kg/min = kg/L \times L/min$

concentration of salt   volume pumped

$$\frac{dm}{dt} = 0 \times 20 - \frac{m}{\text{Volume}} \times 15$$

$$= -\frac{15m}{200+5t}$$

$$= \frac{\cancel{5}\,(-3m)}{\cancel{5}(40+t)}$$

$$\therefore \frac{dm}{dt} = \frac{-3m}{40+t}$$

ii)

$$\frac{dm}{m} = \frac{-3}{40+t}\,dt$$

Integrate both sides

$$\int \frac{dm}{m} = -3\int \frac{1}{40+t}\,dt$$

$$\ln|m| = -3\ln|40+t| + C$$

$$\ln|m| + 3\ln|40+t| = C$$

$$\ln|m(40+t)^3| = C$$

$$m(40+t)^3 = e^C$$

Let $e^C = A$ = constant

at $t=0$ $m=50$

$$\therefore 50(40)^3 = A = 3.2\times10^6$$

$$\therefore m = \frac{3.2\times10^6}{(40+t)^3}$$

iii) Tank overfills when 500L

$$200 + 5t = 500$$
$$5t = 300$$
$$t = 60 \text{ mins}$$

$$\therefore \quad m = \frac{3.2 \times 10^6}{(40+60)^3}$$
$$= 3.2 \text{ kg}$$

13d)

$$A = 2\int_0^{\pi/2} (2\cos x - 2\cos^2 x)\,dx$$

$$2\cos^2 x - 1 = \cos 2x$$
$$\therefore \quad 2\cos^2 x = \cos 2x + 1$$

$$\therefore \quad A = 2\int_0^{\pi/2} (2\cos x - \cos 2x - 1)\,dx$$
$$= 2\left[2\sin x - \frac{\sin 2x}{2} - x\right]_0^{\pi/2}$$
$$= 2\left(2\sin\frac{\pi}{2} - \frac{\sin 2\frac{\pi}{2}}{2} - \frac{\pi}{2} - 2\sin 0 + \frac{\sin 0}{2} - 0\right)$$
$$= 2\left(2 - \frac{0}{2} - \frac{\pi}{2} - 0 + 0 - 0\right)$$
$$= 4 - \pi$$

Question 14

14a) i) $E(X) = np$

$$= 50 \times 0.6$$
$$= 30$$

$$Var(X) = np(1-p)$$
$$= 30(1-0.6)$$
$$= 12$$

ii) $\mu = E(X) = 30$

$\sigma = \sqrt{12} = 3.464$ (3dp)

Z-score for 23

$$Z = \frac{23-30}{3.464} = -2.02 \quad (2dp)$$

Z-score for 27

$$Z = \frac{27-30}{3.464} = -0.87 \quad (2dp)$$

From table

$$P(-2.02 \leq Z \leq -0.87) = 0.2327 - 0.0239$$
$$= 0.2088$$
$$= 20.88\%$$

iii) The experiment result has some weight. It can be improved by increasing the number of trials

14 b)
(i)

$\underset{\sim}{a} = -g\underset{\sim}{j}$

$\underset{\sim}{v} = -\int g\underset{\sim}{j}\, dt$

$= -gt\underset{\sim}{j} + c$

When $t = 0$, $\underset{\sim}{v} = V\cos\theta\,\underset{\sim}{i} + V\sin\theta\,\underset{\sim}{j}$

$\therefore \quad V\cos\theta\,\underset{\sim}{i} + V\sin\theta\,\underset{\sim}{j} = c$

$\underset{\sim}{v} = V\cos\theta\,\underset{\sim}{i} + (V\sin\theta - gt)\underset{\sim}{j}$

$\underset{\sim}{s} = \int\left(V\cos\theta\,\underset{\sim}{i} + (V\sin\theta - gt)\underset{\sim}{j}\right) dt$

$= Vt\cos\theta\,\underset{\sim}{i} + \left(Vt\sin\theta - \frac{gt^2}{2}\right)\underset{\sim}{j} + c$

at $t = 0$, $\underset{\sim}{s} = h\underset{\sim}{j}$

$\therefore \quad h\underset{\sim}{j} = c$

$\therefore \quad \underset{\sim}{s} = Vt\cos\theta\,\underset{\sim}{i} + \left(Vt\sin\theta - \frac{gt^2}{2} + h\right)\underset{\sim}{j}$

as $\theta = 0°$

$\therefore \quad \underset{\sim}{s} = Vt\,\underset{\sim}{i} + \left(h - \frac{1}{2}gt^2\right)\underset{\sim}{j}$

ii) $V = 252 \text{ km/h}$

$= \frac{252 \times 1000}{60 \times 60}$

$= 70 \text{ m/s}$

Given $h = 150 \text{ m}$, $V = 70 \text{ m/s}$, $g = 10 \text{ m/s}^2$, $\theta = 0°$

Using y component of $\underset{\sim}{s}$ when $y = 0$

$$h - \frac{1}{2}gt^2 = 0$$

$$150 - \frac{1}{2} \times 10t^2 = 0$$

$$t^2 = \frac{150}{5}$$

$$t = \pm\sqrt{30}$$

as time is always positive

$t = \sqrt{30}$ seconds

≈ 5.5 seconds for package to reach the sailboat.

iii) Boat drifts at

$1.8 \text{ km/h} = \frac{1800}{60 \times 60}$

$= 0.5 \text{ m/s}$.

In $\sqrt{30}$ seconds

→ boat drifts $0.5 \times \sqrt{30}$ m

→ Let 'd' be the horizontal distance from plane to boat when package is dropped

$\therefore$ package needs to cover

$d + 0.5 \times \sqrt{30}$ meters

In $\sqrt{30}$ seconds package moves x horizontal distance

$$x_{\underset{\sim}{i}} = Vt$$
$$= 70 \times \sqrt{30}$$
$$\doteq 383.4 \quad (1\,dp)$$

$\therefore$ with 40 m of this distance the package can be dropped

$$383.4 - 40 \leq d + 0.5 \times \sqrt{30} \leq 383.4 + 40$$

$$343.4 - 0.5 \times \sqrt{30} \leq d \leq 423.4 - 0.5 \times \sqrt{30}$$

$$340.7 \leq d \leq 420.7$$

$\therefore$ Drop the package between 340.7m and 420.7m from sailboat.

14 c) Given: kite $\rightarrow |AB| = |AD|$, $|BC| = |CD|$

$\overrightarrow{AB} = (b - a)$ $\quad\quad \overrightarrow{DC} = c - d$

$\overrightarrow{BC} = (c - b)$ $\quad\quad \overrightarrow{AD} = d - a$

Prove $\quad \overline{AC} \cdot \overline{BD} = 0$

i.e. $\quad (c - a) \cdot (d - b) = 0$

$$|\vec{AB}|^2 = |\vec{AD}|^2$$
$$(b-a)^2 = (d-a)^2$$
$$b^2 - 2a.b + \cancel{a^2} = d^2 - 2a.d + \cancel{a^2}$$
$$2a.d - 2a.b = d^2 - b^2$$
$$2a(d-b) = d^2 - b^2$$
$$2a\cancel{(d-b)} = \cancel{(d-b)}(d+b)$$
$$2a = d+b \quad \text{—— } 1$$

Similarly $|\vec{BC}|^2 = |\vec{DC}|^2$
$$(c-b)^2 = (c-d)^2$$
$$\cancel{c^2} - 2c.b + b^2 = \cancel{c^2} - 2c.d + d^2$$
$$2c.d - 2c.b = d^2 - b^2$$
$$2c(d-b) = d^2 - b^2$$
$$2c\cancel{(d-b)} = \cancel{(d-b)}(d+b)$$
$$2c = d+b \quad \text{—— } 2$$

Solving 1 & 2
$$2a = 2c$$
$$\therefore a = c \quad \text{—— } 3$$
$$\therefore \quad \overline{AC} \cdot \overline{BD} = (c-a)(d-b)$$
Using 3 $= (c-c)(d-b)$
$= 0$

$\therefore$ Diagonals are perpendicular